# Problem 2.18

An infinite plane slab, of thickness $2d$, carries a uniform volume charge density $\rho$ (Fig. 2.27). Find the electric field, as a function of $y$, where $y = 0$ at the center. Plot $E$ versus $y$, calling $E$ positive when $\mathbf{E}$ points in the $+y$ direction and negative when it points in the $-y$ direction.

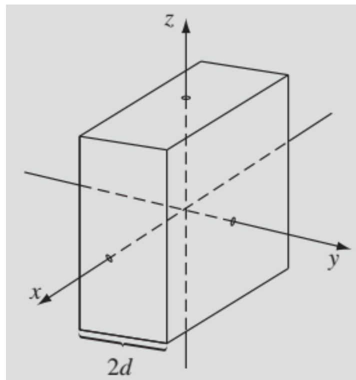


**Fig. 2.27**

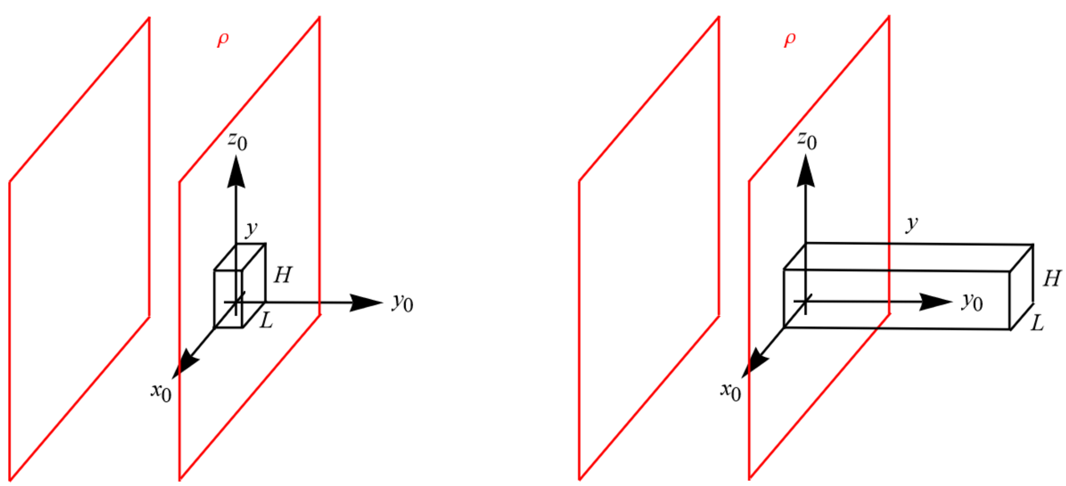
---

### Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the symmetry about the $xz$-plane, the divergence is sufficient. In addition, the electric field is assumed to only depend on $y$ and be along the $y$-axis: $\mathbf{E} = E(y)\hat{\mathbf{y}}$. There's as much charge to the right of the $xz$-plane as there is on the left, so the electric field at $y = 0$ is zero. Integrate both sides over the volume of a (black) rectangular Gaussian surface with length $L$, width $y$, and height $H$. Two cases need to be considered: (i) the case where $0 < y < d$ and (ii) the case where $y > d$.



$0 < y < d$ $\qquad$ $y > d$

The enclosed charge is the product of the charge density with the volume.

$$\int_0^H \int_0^y \int_0^L \nabla \cdot \mathbf{E}\,(dx_0\,dy_0\,dz_0) = \begin{cases} \displaystyle\int_0^H \int_0^y \int_0^L \frac{\rho}{\epsilon_0}\,(dx_0\,dy_0\,dz_0) & \text{if } 0 < y < d \\ \displaystyle\int_0^H \int_0^d \int_0^L \frac{\rho}{\epsilon_0}\,(dx_0\,dy_0\,dz_0) & \text{if } y > d \end{cases}$$

Apply the divergence theorem on the left side.

$$\oiint \mathbf{E} \cdot d\mathbf{S}_0 = \begin{cases} \displaystyle\frac{\rho}{\epsilon_0}\left(\int_0^H dz_0\right)\left(\int_0^y dy_0\right)\left(\int_0^L dx_0\right) & \text{if } 0 < y < d \\ \displaystyle\frac{\rho}{\epsilon_0}\left(\int_0^H dz_0\right)\left(\int_0^d dy_0\right)\left(\int_0^L dx_0\right) & \text{if } y > d \end{cases}$$

Because the electric field only depends on $y$, the surface integral only needs to be calculated on two faces of the box.

$$\int_0^H \int_0^L [E(y_0)\hat{\mathbf{y}}_0 \cdot (\hat{\mathbf{y}}_0\,dx_0\,dz_0)]\bigg|_{y_0=0} + \int_0^H \int_0^L [E(y_0)\hat{\mathbf{y}}_0 \cdot (\hat{\mathbf{y}}_0\,dx_0\,dz_0)]\bigg|_{y_0=y} = \begin{cases} \dfrac{\rho}{\epsilon_0}(HLy) & \text{if } 0 < y < d \\ \dfrac{\rho}{\epsilon_0}(HLd) & \text{if } y > d \end{cases}$$

Evaluate the dot products.

$$\int_0^H \int_0^L E(0)\,dx_0\,dz_0 + \int_0^H \int_0^L E(y)\,dx_0\,dz_0 = \begin{cases} \dfrac{\rho}{\epsilon_0}(HLy) & \text{if } 0 < y < d \\ \dfrac{\rho}{\epsilon_0}(HLd) & \text{if } y > d \end{cases}$$

The electric field is zero on the $y = 0$ face, and the electric field is constant on the $y_0 = y$ face.

$$\int_0^H \int_0^L (0)\,dx_0\,dz_0 + E(y)\int_0^H \int_0^L dx_0\,dz_0 = \begin{cases} \dfrac{\rho}{\epsilon_0}(HLy) & \text{if } 0 < y < d \\ \dfrac{\rho}{\epsilon_0}(HLd) & \text{if } y > d \end{cases}$$

Evaluate the integrals.

$$E(y)(HL) = \begin{cases} \dfrac{\rho}{\epsilon_0}(HLy) & \text{if } 0 < y < d \\ \dfrac{\rho}{\epsilon_0}(HLd) & \text{if } y > d \end{cases}$$

Divide both sides by $HL$ to solve for $E(y)$.

$$E(y) = \begin{cases} \dfrac{\rho}{\epsilon_0}y & \text{if } 0 < y < d \\ \dfrac{\rho}{\epsilon_0}d & \text{if } y > d \end{cases}$$

Due to the symmetry about the $y = 0$ plane and the fact that the electric field is negative for $y < 0$, take the odd extension of $E(y)$ to get the electric field for $-\infty < y < \infty$.

$$E(y) = \begin{cases} -\dfrac{\rho}{\epsilon_0} d & \text{if } y < -d \\[2ex] \dfrac{\rho}{\epsilon_0} y & \text{if } -d < y < d \\[2ex] \dfrac{\rho}{\epsilon_0} d & \text{if } y > d \end{cases}$$

Therefore,

$$\mathbf{E}(y) = \begin{cases} -\dfrac{\rho}{\epsilon_0} d\hat{\mathbf{y}} & \text{if } y < -d \\[2ex] \dfrac{\rho}{\epsilon_0} y\hat{\mathbf{y}} & \text{if } -d < y < d \\[2ex] \dfrac{\rho}{\epsilon_0} d\hat{\mathbf{y}} & \text{if } y > d \end{cases}.$$

Below is a plot of $E(y)$ versus $y$.